Digital signals

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1 Motivating the transition from analog to digital

Various reasons motivate the transition from analog to digital. First, we focus on a mathematical reason. We recall the non-linear equation governing the pendulum studied in the lecture about LTI systems:

$$\forall t \in \mathbb{R} \qquad y''(t) + K\sin(y(t)) = x(t)$$

where $K = \dfrac{mg}{\ell}$, x is the input and y is the output. We have seen that if y only takes small values, i.e. $y(t) \approx 0$ for any $t \in \mathbb{R}$, then we can do the approximation $\sin(y(t)) \approx y(t)$ and linearize the differential equation around the equilibrium $y = 0_{\mathbb{R}}$. However, if y is allowed to take large values, this approximation is not valid anymore and the linearization fails. A possible solution in this case is to replace continuous time $t \in \mathbb{R}$ by discrete time $n \in \mathbb{Z}$. Instead of looking for an exact continuous solution $y(t)$ for any $t \in \mathbb{R}$, we look for an approximate discrete solution $y[n] = y(nT)$ for any $n \in \mathbb{Z}$, with $T > 0$. To solve our problem, we need to introduce a discrete version of first and second derivatives. To remain causal, we define the left-side first derivative:

$$\forall t \in \mathbb{R} \qquad y'(t) = \lim_{h \to 0^+} \frac{y(t) - y(t-h)}{h}$$

Since we only have access to values $y[k] = y(kT)$ at every time step T, we assume that y is locally affine and we write:

$$\forall n \in \mathbb{Z} \qquad y'[n] = y'(nT) \approx \frac{y(nT) - y((n-1)T)}{T} = \frac{y[n] - y[n-1]}{T}$$

Then we define the discrete second derivative by repeating the reasoning:

$$\forall n \in \mathbb{Z} \qquad y''[n] \approx \frac{y'[n] - y'[n-1]}{T} \approx \frac{y[n] - 2y[n-1] + y[n-2]}{T^2}$$

With these approximations, the non-linear pendulum equation becomes:

$$\forall n \in \mathbb{Z} \qquad \frac{y[n] - 2y[n-1] + y[n-2]}{T^2} + K\sin(y[n]) = x[n]$$

which can be rewritten:

$$\forall n \in \mathbb{Z} \qquad y[n] + T^2 K \sin(y[n]) = T^2 x[n] + 2y[n-1] - y[n-2]$$

This **difference equation** is still non-linear but values $y[n]$ can be estimated with numerical methods such as the Newton-Raphson method.

Other technological reasons motivated the transition from analog systems to digital systems. Although analog systems can perform most of the mathematical operations (summation, multiplication, exponential, ...) using operational amplifiers, or act as filters with circuits based on capacitors and inductors, these tasks can be handled by programmable chips which are generally cheaper and more performant.

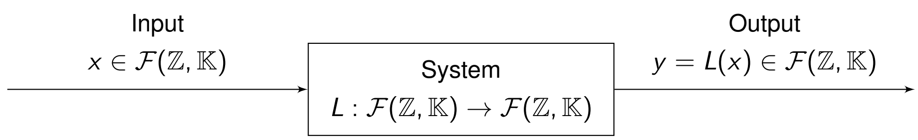
2 Definitions and properties

2.1 Definitions

Definition 2.1 (Digital signal)
A **digital signal** is a mapping from $\mathbb{Z}^m$, with $m \in \mathbb{N}^*$, to $\mathbb{K}$, with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We denote $\mathcal{F}(\mathbb{Z}^m, \mathbb{K})$ the vector space of digital signals. For any signal $x \in \mathcal{F}(\mathbb{Z}^m, \mathbb{K})$, we denote $x[n]$ the sample value of x in $n \in \mathbb{Z}^m$, while we keep the notation $x(t)$ for analog signals.

Remark: In the following lectures, we focus on univariate digital signals, i.e. such that $m = 1$.

Definition 2.2 (Digital system)
A **digital system** is a mapping from $\mathcal{F}(\mathbb{Z}, \mathbb{K})$ to $\mathcal{F}(\mathbb{Z}, \mathbb{K})$. We define the input and the output of a digital system exactly as in the analog case.



Remark: As we have seen in the previous section, digital systems are not governed by differential equations, but by **difference equations**, their digital version. In theses equations, analog derivatives are replaced by digital derivatives:

$$\forall n \in \mathbb{Z} \qquad x'[n] = x[n] - x[n-1] \qquad x''[n] = x[n] - 2x[n-1] + x[n-2]$$

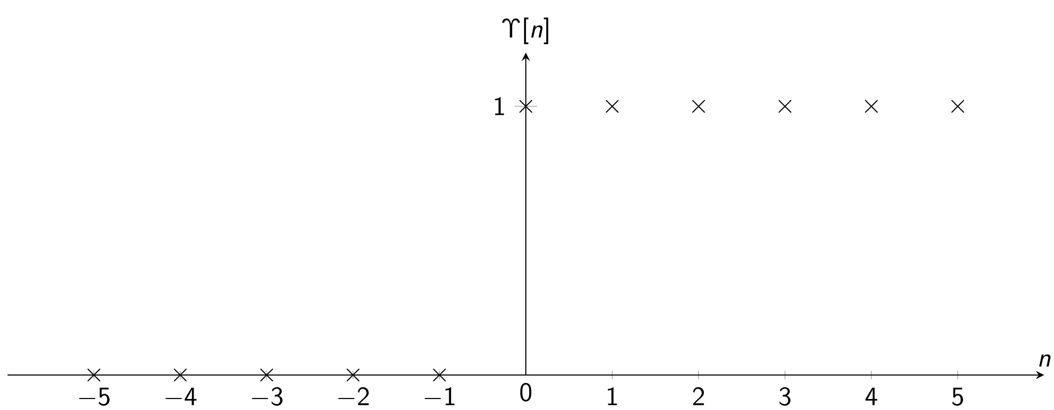
and more generally,

$$\forall m \in \mathbb{N}^* \quad \forall n \in \mathbb{Z} \qquad x^{(m)}[n] = \sum_{k=0}^{m} \binom{m}{k} (-1)^k x[n-k]$$

2.2 Digital Heaviside step function and Dirac delta function

Definition 2.3 (Digital Heaviside step function)
The **digital Heaviside step function** is the signal $\Upsilon \in \mathcal{F}(\mathbb{Z}, \mathbb{R})$ defined by

$$\forall n \in \mathbb{Z} \qquad \Upsilon[n] = \begin{cases} 0 & \text{if } n < 0 \\ 1 & \text{if } n \geq 0 \end{cases}$$

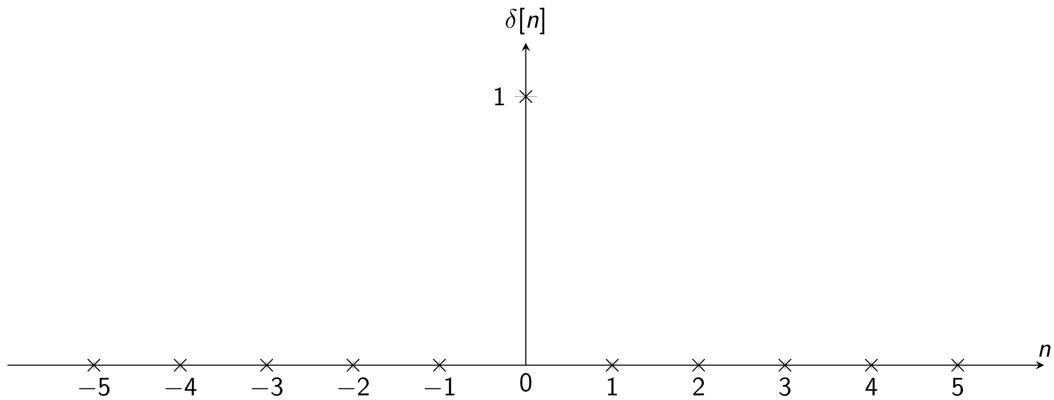


Remark: We can define for any $a \in \mathbb{Z}$ the shifted Heaviside step function $\Upsilon_a[n] = \Upsilon[n-a]$, and for any $\alpha \in \mathbb{K}$, the weighted Heaviside step function $(\alpha\Upsilon)[n] = \alpha\Upsilon[n]$.

Definition 2.4 (Digital Dirac delta function)
The **digital Dirac delta function** is the signal $\delta \in \mathcal{F}(\mathbb{Z}, \mathbb{R})$ defined by

$$\forall n \in \mathbb{Z} \qquad \delta[n] = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}$$

With the Kronecker delta notation, we can write $\delta[n] = \delta_{0,n}$ for any $n \in \mathbb{Z}$.



Remarks:

- We can easily show that for any $n \in \mathbb{Z}$, $\delta[n] = \Upsilon[n] - \Upsilon[n-1]$ so that the digital Dirac delta function is the digital derivative of the digital Heaviside step function. The digital version of the Dirac delta function is much simpler to work with than the analog version.
- We can define for any $a \in \mathbb{Z}$ the shifted digital Dirac delta function $\delta_a[n] = \delta[n-a]$, and for any $\alpha \in \mathbb{K}$, the weighted digital Dirac delta function $(\alpha\delta)[n] = \alpha\delta[n]$.
- The **step response** (resp. the **impulse response**) of a digital system is the output corresponding to the digital Heaviside step function (resp. the digital Dirac delta function) as input.

2.3 LTI systems and convolution

Definition 2.5
Let L be a digital signal from $\mathcal{F}(\mathbb{Z}, \mathbb{K})$ to $\mathcal{F}(\mathbb{Z}, \mathbb{K})$.

- L is a **linear system** if it is a linear mapping:

$$\forall (x_1, x_2) \in \mathcal{F}(\mathbb{Z}, \mathbb{K})^2 \quad L(x_1 + x_2) = L(x_1) + L(x_2) \qquad \forall x \in \mathcal{F}(\mathbb{Z}, \mathbb{K}) \quad \forall \alpha \in \mathbb{K} \qquad L(\alpha x) = \alpha L(x)$$

- For any $a \in \mathbb{Z}$, the **pure delay** or **shifting** system is defined as $\tau_a : x \mapsto x_a$ with $x_a : t \mapsto x[n-a]$. L is a time-invariant system if it commutes with any pure delay i.e. for any $a \in \mathbb{Z}$, $L \circ \tau_a = \tau_a \circ L$.
- Digital **linear time-invariant** (LTI) systems satisfy both properties (i) and (ii).
- L is a **causal system** if, for any input x and any $n \in \mathbb{Z}$, the instantaneous output $L(x)[n]$ only depends on prior values of x, i.e. values $x[n-k]$ for $k \geq 0$.

Definition 2.6 (Digital convolution)
Digital convolution $*$ defines a product in $\mathcal{F}(\mathbb{Z}, \mathbb{K})$ by:

$$\forall (x, y) \in \mathcal{F}(\mathbb{Z}, \mathbb{K})^2 \qquad \forall n \in \mathbb{Z} \qquad (x * y)[n] = \sum_{k=-\infty}^{+\infty} x[k]y[n-k]$$

Remark: In the next lecture, we will see how digital convolution can be deduced from analog convolution.

Definition 2.7 (Digital convolution system)
A **digital convolution system** is a mapping from $\mathcal{F}(\mathbb{Z}, \mathbb{K})$ to $\mathcal{F}(\mathbb{Z}, \mathbb{K})$ of the form $x \mapsto x * h$, where h is a given function.

Remarks:

- As in the analog case, the digital LTI systems are exactly the digital convolution systems.
- Let L be a digital LTI system with impulse response h. Let an input signal x and the corresponding output $y = L(x) = x * h$. From the definition of convolution:

$$\forall n \in \mathbb{Z} \qquad y[n] = (x * h)[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = \sum_{k=-\infty}^{+\infty} x[n-k]h[k]$$

  System L is causal if the instantaneous output $y[n]$ only depends on prior input $x[n-k]$ for $k \geq 0$. It implies the following sufficient condition:

  If for any $n < 0$, $h[n] = 0$, then L is a causal system.

Proposition 2.1
Convolution satisfies the following properties:

(i) it is bilinear: for any signals x, y and z, and any scalar $\alpha \in \mathbb{K}$, $x * (y + \alpha z) = (x * y) + \alpha(x * z)$, same for the first component;

(ii) it is associative and commutative: for any x, y and z, $x * (y * z) = (x * y) * z$ and $x * y = y * x$;

(iii) it commutes with pure delays: for any $a \in \mathbb{Z}$, $\tau_a(x * y) = \tau_a(x) * y = x * \tau_a(y)$;

(iv) summation: for any signals x and y,

$$\sum_{n=-\infty}^{+\infty} (x * y)[n] = \left(\sum_{n=-\infty}^{+\infty} x[n]\right)\left(\sum_{n=-\infty}^{+\infty} y[n]\right)$$

(v) digital differentiation: for any digital signals x and y, $(x * y)' = x' * y = x * y'$.

Proof : These properties are deduced from their analog version, as shown in the next lecture. ■

2.4 Scalar product and correlation

In this subsection, we restrict to the subspace $L^2(\mathbb{Z}, \mathbb{K})$ of signals in $\mathcal{F}(\mathbb{Z}, \mathbb{K})$ which are square summable.

Definition 2.8 (Scalar product over $\mathbf{L}^2(\mathbb{Z},\mathbb{R})$, Hermitian product over $\mathbf{L}^2(\mathbb{Z},\mathbb{C})$, energy)
We define a **scalar product** over $L^2(\mathbb{Z},\mathbb{R})$ by

$$\forall (x,y) \in L^2(\mathbb{Z},\mathbb{R})^2 \qquad \langle x, y\rangle = \sum_{n=-\infty}^{+\infty} x[n]y[n]$$

We define a **Hermitian product** over $L^2(\mathbb{Z},\mathbb{C})$ by

$$\forall (x,y) \in L^2(\mathbb{Z},\mathbb{C})^2 \qquad \langle x, y\rangle = \sum_{n=-\infty}^{+\infty} x[n]\overline{y[n]}$$

From these products, we can define the norm of a signal, from which we introduce the **energy**:

$$\forall x \in L^2(\mathbb{Z},\mathbb{K}) \qquad E(x) = \|x\|^2 = \langle x, x\rangle$$

i.e.

$$\forall x \in L^2(\mathbb{Z},\mathbb{R}) \quad E(x) = \sum_{n=-\infty}^{+\infty} x[n]^2 \qquad \forall x \in L^2(\mathbb{R},\mathbb{C}) \quad E(x) = \sum_{n=-\infty}^{+\infty} |x[n]|^2$$

Definition 2.9 (Average power)
The **average power** of a digital signal $x \in \mathcal{F}(\mathbb{Z},\mathbb{K})$ is the real number:

$$P(x) = \lim_{N\to+\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

Definition 2.10 (Cross-correlation, autocorrelation)
Let x and y be two signals of $L^2(\mathbb{Z},\mathbb{K})$. The **cross-correlation** is the function $\gamma_{xy} : \mathbb{Z} \to \mathbb{K}$ defined by:

$$\forall n \in \mathbb{Z} \qquad \gamma_{xy}[n] = \langle x, \tau_n(y)\rangle = \sum_{n=-\infty}^{+\infty} x[k]y^*[k-n]$$

The **autocorrelation** of a signal $x \in L^2(\mathbb{Z},\mathbb{K})$ is the cross-correlation with itself, i.e.

$$\forall n \in \mathbb{Z} \qquad \gamma_x[n] = \gamma_{xx}[n] = \langle x, \tau_n(x)\rangle = \sum_{n=-\infty}^{+\infty} x[k]x^*[k-n]$$

Proposition 2.2
We have the following properties:

(i) For any two signals x and y, cross-correlation satisfies the inequality:

$$\forall n \in \mathbb{Z} \qquad |\gamma_{xy}[n]| \leq \sqrt{E(x)E(y)}$$

In particular, for any signal x, the absolute value of autocorrelation γ_x reaches its maximum $E(x)$ in 0.

(ii) Autocorrelation satisfies the following symmetry property: for any signal x, for any $n \in \mathbb{Z}$, $\gamma_x[-n] = \gamma_x^*[n]$.

PROOF : These properties are proved like their analog versions. ■

2.5 Periodic signals

Definition 2.11 (Periodic signal, fundamental period)
A digital signal $x \in \mathcal{F}(\mathbb{Z}, \mathbb{K})$ is **periodic** if there exists $N \in \mathbb{N}^*$ such that for any $n \in \mathbb{Z}$, $x[n+N] = x[n]$. The smallest integer $N \in \mathbb{N}^*$ such that $x[n+N] = x[n]$ for any $n \in \mathbb{Z}$ is the **fundamental period**. We denote $\mathcal{F}_N(\mathbb{Z}, \mathbb{K})$ the subspace of periodic signals with period N.

Definition 2.12 (Fundamental frequency, fundamental impulse)
Let $x \in \mathcal{F}_N(\mathbb{R}, \mathbb{K})$ be a digital periodic signal with period N. The **fundamental frequency** of x is the number $f = \frac{1}{N}$, and the **fundamental impulse** of x is the number $\omega = 2\pi f = \frac{2\pi}{N}$.

Remark: As for the analog case, if a digital periodic signal with period N is the input of a digital LTI system, the corresponding output is also a digital periodic signal with period N.

We can prove as in the analog case that any non-zero digital periodic signal with period N has infinite energy. However, the average power of this periodic signal is:

$$P(x) = \lim_{M \to +\infty} \frac{1}{2M+1} \sum_{n=-M}^{M} |x[n]|^2 = \frac{1}{N} \sum_{n=0}^{N-1} |x[n]|^2$$

We define the subspace of $\mathcal{F}_N(\mathbb{Z}, \mathbb{K})$ of digital periodic signals with period N which are locally summable, i.e. which have finite average power.

Definition 2.13
We define a **scalar** / **Hermitian product** over $\mathcal{F}_N(\mathbb{Z}, \mathbb{K})$ by

$$\forall (x, y) \in \mathcal{F}_N(\mathbb{Z}, \mathbb{K})^2 \qquad \langle x, y \rangle_N = \frac{1}{N} \sum_{n=0}^{N-1} x[n] y^*[n]$$

From this scalar / Hermitian product, we can define the norm of a signal x to which we can connect the average power of the signal:

$$\forall x \in \mathcal{F}_N(\mathbb{Z}, \mathbb{K})^2 \qquad P(x) = \|x\|_N^2 = \langle x, x \rangle_N$$

i.e.

$$\forall x \in \mathcal{F}_N(\mathbb{Z}, \mathbb{K}) \qquad P(x) = \frac{1}{N} \sum_{n=0}^{N-1} |x[n]|^2$$

Remark: We can redefine cross-correlation and autocorrelation based on this new product:

$$\gamma_{xy}[n] = \frac{1}{N} \sum_{n=0}^{N-1} x[k] y^*[k-n] \qquad \text{and} \qquad \gamma_x[n] = \frac{1}{N} \sum_{n=0}^{N-1} x[k] x^*[k-n]$$

Proposition 2.3
The autocorrelation of a digital periodic signal with period N is also periodic with period N.

PROOF : The proof is identical to the analog one. ■

Definition 2.14 (Digital circular convolution)
The **digital circular convolution** $\otimes$ is a product in $\mathcal{F}_N(\mathbb{Z}, \mathbb{K})$ defined by:

$$\forall (x, y) \in \mathcal{F}_N(\mathbb{Z}, \mathbb{K})^2 \qquad \forall n \in \mathbb{Z} \qquad (x \otimes y)[n] = \frac{1}{N} \sum_{k=0}^{N-1} x[k]y[n-k]$$

Remark: The circular convolution of two digital periodic signals x and y with period N is also periodic with period N.